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Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes should not exceed 2500 words (where a figure or table counts as 200 words). Following informal review by the Editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

State-Space Formula for the Two-Input/Two-Output Coupling Numerator

Kevin Blankinship*
The Boeing Company, Oklahoma City, Oklahoma 73135

Introduction

BEFORE the development of state-space methods of control law design and analysis, coupling numerators were introduced as a means for analysis and design of multi-input/multi-output control laws.[1] Coupling numerators have proven to be highly useful for accounting for the zeros of individual single-input/single-output transfer functions migrate due to closures of different loops, for example, how the roll/aileron zeros are affected by a yaw damper. Although superceded by the numerous design methodologies that have been developed over the years, coupling numerators have proven useful in gaining insight into how one feedback loop affects the dynamics of other inputs and outputs of the system. It is surprising that no attempt was made to incorporate coupling numerators into linear state-space control theory. Because of properties such as that the roots of the coupling numerator are the transmission zeros,[2,3] this would seem appropriate. A contribution to the rectification of that oversight is the subject of this Engineering Note.

Statement of the Problem

The linear equations of motion are assumed to have the form

$$\dot{\boldsymbol{x}} = A\boldsymbol{x} + B\boldsymbol{u} \tag{1}$$

$$\boldsymbol{y} = C\boldsymbol{x} + D\boldsymbol{u} \tag{2}$$

where $\boldsymbol{x}$ is the state vector, $\boldsymbol{u}$ is the input, and $\boldsymbol{y}$ is the output of the system.

The transfer matrix of this system is given by the formula

$$G(s) = C[sI - A]^{-1}B + D \tag{3}$$

that, on solution, has the form of a numerator polynomial matrix $N(s)$ divided by the characteristic polynomial of A, denoted by $\psi(s)$

$$G(s) = N(s)/\psi(s) \tag{4}$$

The formula for the two-input/two-output coupling numerator can then be stated

$$N^{y_i y_p}_{u_j u_r}(s) = [1/\psi(s)]\left\{N^{y_i}_{u_j}(s)N^{y_p}_{u_r}(s) - N^{y_i}_{u_r}(s)N^{y_p}_{u_j}(s)\right\} \tag{5}$$

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*Engineer 4, Navigation Group, B-1B Program; Kevin.G.Blankinship@boeing.com. Member AIAA.

where $N^{y_i}_{u_j}(s)$ denotes the element in the ith row and jth column of $N(s)$.

The goal of this Engineering Note is to express $N^{y_i y_p}_{u_j u_r}(s)$ in terms of a state realization of the equations of motion. The approach taken to the derivation of this formula makes use of the canonical form realization of the state equations. Because this realization is a simple transformation of coordinates, the form of the formula for $G(s)$ is unaffected.

Formulation

Define the matrix P to be the matrix of eigenvectors of A and Λ to be a diagonal matrix of eigenvalues of A. These matrices can be defined by the following property:

$$AP = P\Lambda \tag{6}$$

where

$$\Lambda = \mathrm{diag}\{s_k;\ k = 1, 2, \ldots, n\} \tag{7}$$

By reparameterization of the state vector with

$$\boldsymbol{q} = P^{-1}\boldsymbol{x} \tag{8}$$

the state equations using this new realization will have the form

$$\dot{\boldsymbol{q}} = \Lambda\boldsymbol{q} + B_m\boldsymbol{u} \tag{9}$$

$$\boldsymbol{y} = C_m\boldsymbol{q} + D\boldsymbol{u} \tag{10}$$

where

$$B_m = P^{-1}B \tag{11}$$

$$C_m = CP \tag{12}$$

The equation for the transfer matrix becomes

$$G(s) = C_m[sI - \Lambda]^{-1}B_m + D \tag{13}$$

The inverse $[sI - \Lambda]^{-1}$ is written as[4]

$$[sI - \Lambda]^{-1} = R(s)/\psi(s) \tag{14}$$

where $R(s)$ is the resolvent matrix. When canonical coordinates are used, $\Lambda(s)$ is diagonal, which causes $R(s)$ to also be diagonal with elements

$$R(s) = \mathrm{diag}\{[\psi(s)/(s - s_k)];\ k = 1, 2, \ldots, n\} \tag{15}$$

The formula for the numerator matrix becomes

$$N(s) = C_m R(s) B_m + D\psi(s) \tag{16}$$

The formula for $N^{y_i}_{u_j}(s)$ is then

$$N^{y_i}_{u_j}(s) = \sum_{k=1}^{n} R_k(s) C_{m_{ik}} B_{m_{kj}} + D_{ij}\psi(s) \tag{17}$$

Substitution of Eq. (17) into Eq. (5) gives the expression

$$N_{u_j u_r}^{y_i y_p}(s) = \frac{1}{\psi(s)} \left\{ \begin{aligned} &\sum_{k=1}^{n} R_k(s) C_{m_{ik}} B_{m_{kj}} \sum_{h=1}^{n} R_h(s) C_{m_{ph}} B_{m_{hr}} \\ &- \sum_{k=1}^{n} R_k(s) C_{m_{ik}} B_{m_{kr}} \sum_{h=1}^{n} R_h(s) C_{m_{ph}} B_{m_{hj}} \\ &+ D_{ij}\psi(s) \sum_{k=1}^{n} R_k(s) C_{m_{pk}} B_{m_{kr}} \\ &+ D_{pr}\psi(s) \sum_{k=1}^{n} R_k(s) C_{m_{ik}} B_{m_{kj}} \\ &- D_{ir}\psi(s) \sum_{k=1}^{n} R_k(s) C_{m_{pk}} B_{m_{kj}} \\ &- D_{pj}\psi(s) \sum_{k=1}^{n} R_k(s) C_{m_{ik}} B_{m_{kr}} \\ &+ (D_{ij}D_{pr} - D_{ir}D_{pj})\psi^2(s) \end{aligned} \right\} \tag{18}$$

Next, combine summations to get

$$N_{u_j u_r}^{y_i y_p}(s) = \frac{1}{\psi(s)} \left\{ \begin{aligned} &\sum_{k=1}^{n}\sum_{h=1}^{n} R_k(s) R_h(s) \big(C_{m_{ik}} B_{m_{kj}} C_{m_{ph}} B_{m_{hr}} - C_{m_{ik}} B_{m_{kr}} C_{m_{ph}} B_{m_{hj}} \big) \\ &+ \psi(s) \sum_{k=1}^{n} R_k(s) \big(D_{ij} C_{m_{pk}} B_{m_{kr}} + D_{pr} C_{m_{ik}} B_{m_{kj}} - D_{ir} C_{m_{pk}} B_{m_{kj}} - D_{pj} C_{m_{ik}} B_{m_{kr}} \big) \\ &+ (D_{ij}D_{pr} - D_{ir}D_{pj})\psi^2(s) \end{aligned} \right\} \tag{19}$$

However, recall from Eq. (15) that

$$R_k(s) = \psi(s)/(s - s_k) \tag{20}$$

Substitution into Eq. (19) allows us to cancel the characteristic polynomial in the denominator to get

$$\begin{aligned} N_{u_j u_r}^{y_i y_p}(s) = &\sum_{k=1}^{n}\sum_{h=k}^{n} \frac{\psi(s)}{(s - s_k)(s - s_h)} \big(C_{m_{ik}} B_{m_{kj}} C_{m_{ph}} B_{m_{hr}} \\ &- C_{m_{ik}} B_{m_{kr}} C_{m_{ph}} B_{m_{hj}} \big) + \sum_{k=1}^{n} \frac{\psi(s)}{(s - s_k)} \big(D_{ij} C_{m_{pk}} B_{m_{kr}} \\ &+ D_{pr} C_{m_{ik}} B_{m_{kj}} - D_{ir} C_{m_{pk}} B_{m_{kj}} - D_{pj} C_{m_{ik}} B_{m_{kr}} \big) \\ &+ (D_{ij}D_{pr} - D_{ir}D_{pj})\psi(s) \end{aligned} \tag{21}$$

However, this turns out to be a polynomial function of s. The reason that this is so is because the rational function terms in this equation are exactly divisible by their denominators. There is one exception, and that is when $h = k$ in the first summation. However, its coefficient is zero, and so this term can be ignored. From symmetry of the coefficients, the double sum can be reindexed to give the following polynomial formula:

$$\begin{aligned} N_{u_j u_r}^{y_i y_p}(s) = 2 &\sum_{k=1}^{n}\sum_{h=k+1}^{n} \left(\prod_{\substack{t=1 \\ t \neq k,h}}^{n} (s - s_t) \right) \big(C_{m_{ik}} B_{m_{kj}} C_{m_{ph}} B_{m_{hr}} \\ &- C_{m_{ik}} B_{m_{kr}} C_{m_{ph}} B_{m_{hj}} \big) + \sum_{k=1}^{n} \left(\prod_{\substack{t=1 \\ t \neq k}}^{n} (s - s_t) \right) \big(D_{ij} C_{m_{pk}} B_{m_{kr}} \\ &+ D_{pr} C_{m_{ik}} B_{m_{kj}} - D_{ir} C_{m_{pk}} B_{m_{kj}} - D_{pj} C_{m_{ik}} B_{m_{kr}} \big) \\ &+ (D_{ij}D_{pr} - D_{ir}D_{pj})\psi(s) \end{aligned} \tag{22}$$

Equation (22) is a polynomial formula for the coupling numerator in terms of the canonical form of the state equations of motion. Because no cancellations with a denominator are required with this formula, it is amenable to straightforward implementation in computer code.

Conclusions

A coupling numerator formula has been derived that gives a state-space expression for the two-input/two-output coupling numerator in a form that can be readily implemented in code. The derivation of this formula explains the observation that the characteristic polynomial cancels out of the coupling numerator, always giving a polynomial for the coupling numerator as opposed to a rational function. By a similar derivation, one could also obtain formulas for N-input/N-output coupling numerators, given N, although formulas for $N = 2$ and 3 would probably suffice for most applications.

References

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